

Noise Threshold of Quantum Supremacy

Keisuke Fujii^{1,2}

¹*Photon Science Center, Graduate School of Engineering,
The University of Tokyo, 2-11-16 Yayoi, Bunkyo-ku, Tokyo 113-8656, Japan*

²*JST, PRESTO, 4-1-8 Honcho, Kawaguchi, Saitama, 332-0012, Japan*

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Demonstrating quantum supremacy, a complexity-guaranteed quantum advantage against over the best classical algorithms by using less universal quantum devices, is an important near-term milestone for quantum information processing. Here we develop a threshold theorem for quantum supremacy with noisy quantum circuits in the pre-threshold region, where quantum error correction does not work directly. We show that, even in such a region, we can virtually simulate quantum error correction by postselection. This allows us to show that the output sampled from the noisy quantum circuits (without postselection) cannot be simulated efficiently by classical computers based on a stable complexity theoretical conjecture, i.e., non-collapse of the polynomial hierarchy. By applying this to fault-tolerant quantum computation with the surface codes, we obtain the threshold value 2.84% for quantum supremacy, which is much higher than the standard threshold 0.75% for universal fault-tolerant quantum computation with the same circuit-level noise model. Moreover, contrast to the standard noise threshold, the origin of quantum supremacy in noisy quantum circuits is quite clear; the threshold is determined purely by the threshold of magic state distillation, which is essential to gain a quantum advantage.

I. INTRODUCTION

One of the most important goals of quantum information processing is to demonstrate quantum speedup over the best classical algorithms, namely quantum supremacy [1–3] to disprove the extended Church-Turing thesis, saying that any efficient computation by a realistic physical device can be efficiently simulated by a probabilistic Turing machine. For example, if we have an ideal universal quantum computer, Shor’s factorization algorithm [4] allows us to demonstrate a super-polynomial quantum speedup over the best known classical algorithms. Moreover, the threshold theorem for fault-tolerant quantum computation guarantees that an ideal universal quantum computer can be constructed from realistic quantum physical devices being subject to physically realistic imperfections [5–8]. Therefore, massive efforts have been paid for experimental realizations of fault-tolerant quantum error correction [9–15].

Recently intermediate models of quantum computation are attracting much attention to show quantum supremacy in experimentally feasible settings. They are not rich enough to perform universal quantum computation, but still provide nontrivial outputs, which could not be simulated efficiently by classical computers. BosonSampling with free bosons [3], instantaneous quantum polynomial-time computation (IQP) with commuting quantum circuits [16–21] (see also depth-four circuits [22]), highly mixed deterministic quantum computation with one-clean qubit (DQC1) [23–25] are examples of those. All of them are experimentally well motivated as linear optical quantum computations [26], quench dynamics with Ising interactions [35, 36], and NMR ensemble quantum computation [23]. Specifically, it has been shown that if the output of these intermediate models are sampled efficiently by a classical computer, the poly-

nomial hierarchy (PH), a generalization of NP (nondeterministic polynomial-time computation) to oracle machines, collapses to the third (or second [25]) level [3, 17]. The collapse of the PH is thought to be highly implausible (for example, $P=NP$ implies that a complete collapse of the PH), and hence classical simulation of the intermediate models is also thought to be hard. Based on this understanding, several BosonSampling experiments have been performed already [27–34].

Here we consider quantum supremacy of noisy quantum circuits in the pre-threshold region, where the noise strength is much higher than the standard threshold of universal fault-tolerant quantum computation. Hence, we cannot employ quantum error correction directly. Then we ask whether or not there is still surviving quantum supremacy in such noisy quantum circuits. The motivation of this question is threefold. (i) There have been several noise thresholds above which such noisy quantum circuits are classically simulatable exactly. However, they are assuming ideal stabilizer operations [37, 38], or there still be a large gap to the threshold of universal fault-tolerant quantum computation [39–41]. The characterization of the intermediate pre-threshold region, where the standard quantum error correction does not work, has been fully open for a long time. (ii) The hardness proofs of the existing intermediate models require the sampling with constant multiplicative errors or constant additive errors with l_1 -norm [3, 17, 18, 20, 21]. These notions of approximation are quite sensitive to noise. If noisy quantum circuits of a constant noise strength are employed, these criteria cannot be achieved directly. Thus, whether or not such noisy quantum circuits themselves can exhibit quantum supremacy has been an important open problem (see also Ref. [19]). (iii) Nowadays, it is becoming possible to operate scalable quantum devices such as superconducting qubits around the standard

threshold of universal fault-tolerant quantum computation [9–15]. Therefore, a complexity-guaranteed criterion of quantum supremacy for such noisy quantum devices in the pre-threshold region is highly demanded in the experiments [42].

To address these issues, we derive a threshold theorem for quantum supremacy with noisy quantum circuits: if the noise strength is lower than a certain threshold value, the output of such noisy quantum circuits cannot be simulated efficiently by classical computers unless the PH hierarchy collapses to the third level. Contrast to the existing intermediate models [3, 16–21, 24, 25], in this work we employ quantum circuits generated from a universal set of gates, but being subject to rather strong noise, which makes the system less universal. To show hardness of classical simulation, we employ the postselection argument [17, 43]. That is, we show that noisy quantum circuits can solve a postBQP-complete, or equivalently PP-complete problem under postselection. To this end, we first show that simulation of an arbitrary two-qubit output of universal quantum computation with an exponentially small additive error is enough to obtain the hardness result by postselection. Then, we show that noisy quantum circuits can achieve it by virtue of postselection, where any outcomes of syndrome measurements suggesting existence of errors are discarded by postselection. This allows us to simulate universal quantum computation with an exponentially small additive error even in the pre-threshold region, where the standard quantum error correction is not available. As a technical point of view, the standard threshold theorem for universal fault-tolerant quantum computation cannot be employed in the above argument. This is because, in the postselection argument, we have to treat a conditional probability distribution conditioned on an exponentially rare postselection event. Therefore we derive a postselected version of the threshold theorem.

The important implications of the postselected version of the threshold theorem are as follows. First, while the raw outputs of noisy quantum circuits cannot satisfy the criteria for quantum supremacy directly, the logical output after an appropriate classical processing can exhibit quantum supremacy by virtue of quantum error correction, which is virtually simulated by using postselection. Second, the threshold value for quantum supremacy is much higher than that of universal fault-tolerant quantum computation. This is because, we can discard any erroneous events suggested by the non-zero error syndromes. Therefore the threshold is given not by the error correction property of quantum error correction codes but by the error detection property. Third, by virtue of the above effect, the origin of quantum supremacy in noisy quantum circuits is quite clear; it is determined by distillability of the magic state [44, 45]. More precisely, in the standard construction of fault-tolerant quantum computation, error correction for Clifford gates limits the threshold, while the threshold of magic state distillation, where error detection is employed, is much higher [46–

49]. In the case of quantum supremacy, we can employ error detection for both Clifford gates and magic state distillation, and hence distillability of the magic state determines the threshold of quantum supremacy. This is quite reasonable, since magic state distillation for non-Clifford gates is essential for make quantum computation classically intractable [37, 38, 40, 50]. Finally, we calculate the threshold value of fault-tolerant quantum computation using the surface code on the two-dimensional array of qubits [48, 49] to obtain a practically meaningful threshold of quantum supremacy. While we cannot simulate postselected events numerically and hence employ an analytical treatment, which underestimates the threshold, the resultant threshold value 2.84% is rather high compared to the standard threshold 0.75% [46, 47] under the same circuit-based depolarizing noise model. This level of noise is within reach of current state-of-the-art experiments in scalable superconducting qubit systems [9–15], and hence it would be possible to observe complexity-guaranteed quantum supremacy with noisy quantum circuits in the near future.

The rest of the paper is organized as follows. In Sec. II, we briefly review the standard threshold theorem of universal fault-tolerant quantum computation. In Sec. III, we construct the postselected threshold theorem of quantum supremacy with noisy quantum circuits. In Sec. IV, we perform a case study for concatenated quantum computation. In Sec. V, we apply the postselected threshold theorem to topological fault-tolerant quantum computation with the surface codes to derive the practical threshold value of quantum supremacy with the circuit-level noise. Section VI is devoted to conclusion and discussion.

II. STANDARD THRESHOLD THEOREM

Let us briefly review the standard threshold theorem of fault-tolerant quantum computation [8, 51]. The ideal unitary gates in a fault-tolerant universal quantum computation is denoted by

$$\mathcal{U} = \prod_k \mathcal{U}_k, \quad (1)$$

where \mathcal{U}_k is the k th unitary gate and is chosen from a universal set of gates. Quantum error correction employs both projective measurements and adaptive operations based on the measurement outcomes. Here, for simplicity, all these operations including classical processing are denoted by unitary gates coherently and included in \mathcal{U} . Then, the probability distribution of the final output of quantum computation is denoted in terms of a projective operator P_x ($x \in \{0, 1\}$) of the final readout as

$$p(x) = \text{Tr}[P_x \mathcal{U}(\rho_{\text{ini}})], \quad (2)$$

where ρ_{ini} is the initial state. In order to take noise into account, each ideal unitary gate \mathcal{U}_k is replaced with

a noisy one $\mathcal{N}_k \mathcal{U}_k$, where \mathcal{N}_k is a completely-positive-trace-preserving (CPTP) map representing the imperfection. Here we assume the noise is local and Markovian. We define a noise strength $\epsilon_k \equiv \|\mathcal{I} - \mathcal{N}_k\|_\diamond$ for each \mathcal{N}_k , where $\|\cdot\|_\diamond$ is the diamond norm for super-operators [52]. Then the noisy version of the unitary gates is given by $\mathcal{U}^{\text{noisy}} \equiv \prod_k (\mathcal{N}_k \mathcal{U}_k)$. Note that imperfections on the initial state ρ_{ini} and the final measurement P_x are also taken as imperfections on unitary gates at certain locations (this is always possible since we can insert an identity gate after the state preparation and before the measurement). Note also that the unitary gates corresponding to the classical processing are assumed to be noise-free, since they are introduced just to simplify the argument. They are implicitly omitted in the following argument. The probability distribution of the final output under the noise is given by

$$p^{\text{noisy}}(x) = \text{Tr}[P_x \mathcal{U}^{\text{noisy}}(\rho_{\text{ini}})]. \quad (3)$$

To argue fault-tolerance, we decompose the noise map \mathcal{N}_k into ideal and noisy parts as follows:

$$\mathcal{N}_k = (1 - \epsilon_k)\mathcal{I} + \mathcal{E}_k \quad (4)$$

where we should note that the noisy part \mathcal{E}_k (with $\|\mathcal{E}_k\|_\diamond \leq 2\epsilon_k$) is not always a CPTP map. Then we expand $\mathcal{U}^{\text{noisy}}$ as a summation over possible paths

$$\mathcal{U}^{\text{noisy}} = \prod_k \{[(1 - \epsilon_k)\mathcal{I} + \mathcal{E}_k]\mathcal{U}_k\} \quad (5)$$

$$= \sum_{\{\eta_k\}} \prod_k \left\{ [(1 - \epsilon_k)\mathcal{I}]^{1-\eta_k} \mathcal{E}_k^{\eta_k} \mathcal{U}_k \right\}, \quad (6)$$

where $\eta_k \in \{0, 1\}$ and $\sum_{\{\eta_k\}}$ is taken over all paths. Now we decompose these paths into sparse and faulty set of paths in such a way that the operator in the sparse set never change the final probability distribution:

$$p(x) \propto \text{Tr} \left[P_x \sum_{\{\eta_k\}|\text{sparse}} \prod_k \left\{ [(1 - \epsilon_k)\mathcal{I}]^{1-\eta_k} \mathcal{E}_k^{\eta_k} \mathcal{U}_k \right\} \rho_{\text{ini}} \right] \quad (7)$$

$$\equiv \alpha p(x). \quad (8)$$

The faulty set is defined as the complement of the sparse set. The sparse and faulty operators (not a density operator) are defined as follows:

$$\rho_{\text{faulty}} \equiv \sum_{\{\eta_k\}|\text{faulty}} \prod_k \left\{ [(1 - \epsilon_k)\mathcal{I}]^{1-\eta_k} \mathcal{E}_k^{\eta_k} \mathcal{U}_k \right\} \rho_{\text{ini}}, \quad (9)$$

$$\rho_{\text{sparse}} \equiv \sum_{\{\eta_k\}|\text{sparse}} \prod_k \left\{ [(1 - \epsilon_k)\mathcal{I}]^{1-\eta_k} \mathcal{E}_k^{\eta_k} \mathcal{U}_k \right\} \rho_{\text{ini}} \quad (10)$$

The error of the probability distribution of the final output is measured by l_1 -norm,

$$\|p(x) - p^{\text{noisy}}(x)\|_{l_1} = \|(1 - \alpha)p(x) + \text{Tr}[P_x \rho_{\text{faulty}}]\|_{l_1}. \quad (11)$$

Since

$$1 = \sum_x \text{Tr}[P_x \mathcal{U}^{\text{noisy}}(\rho_{\text{ini}})] \quad (12)$$

$$= \alpha + \text{Tr}[\rho_{\text{faulty}}], \quad (13)$$

we have $1 - \alpha = \text{Tr}[\rho_{\text{faulty}}]$. Therefore, the error is bounded by

$$\|p(x) - p^{\text{noisy}}(x)\|_{l_1} \leq (1 - \alpha)\|p(x)\|_{l_1} + \|\text{Tr}[P_x \rho_{\text{faulty}}]\|_{l_1}, \quad (14)$$

$$\leq 2\|\rho_{\text{faulty}}\|_1 \quad (15)$$

$$\leq 2 \prod_k (1 - \epsilon_k) \sum_{\{\eta_k\}|\text{faulty}} \left(\frac{2\epsilon_k}{1 - \epsilon_k} \right)^{\eta_k}, \quad (16)$$

where $\|\cdot\|_1$ indicates the operator 1-norm, and we used that $\|\mathcal{E}_k\|_\diamond < 2\epsilon_k$ and $\|\mathcal{U}_k\|_\diamond = 1$. If the system is designed fault-tolerantly, and if the noise strength ϵ_k is smaller than a certain threshold value, then the r.h.s. of Eq. (16) is upper-bounded by an arbitrarily exponentially small value, which we call the standard threshold theorem.

III. POSTSELECTED THRESHOLD THEOREM

Next we consider the case with postselection. In this case, we introduce another two measurement ports corresponding to y and z ($\in \{0, 1\}$). The variable y is employed as a postselection register for postBQP=PP argument [43]. The variable z is used to postselect the events where no syndrome measurement suggests an occurrence of an error. Then we have

$$p(x, y, z) = \text{Tr}[P_{x,y} Q_z \mathcal{U}(\rho_{\text{ini}})], \quad (17)$$

$$p^{\text{noisy}}(x, y, z) = \text{Tr}[P_{x,y} Q_z \mathcal{U}^{\text{noisy}}(\rho_{\text{ini}})], \quad (18)$$

where $P_{x,y}$ and Q_z are projectors corresponding to (x, y) and z , respectively. Since, in the ideal case with \mathcal{U} , z is always zero, we have $p(x, y, 0) \equiv \bar{p}(x, y)$ and $p(x, y, 1) = 0$. Now our goal here is to simulate $\bar{p}(x, y)$ by using postselected noisy quantum computation $p^{\text{noisy}}(x, y|z=0)$ with an exponentially small additive error. Similarly to the standard threshold theorem, we evaluate the error Δ between $\bar{p}(x, y)$ and $p^{\text{noisy}}(x, y|z=0)$:

$$\Delta \equiv \|\bar{p}(x, y) - p^{\text{noisy}}(x, y|z=0)\|_{l_1} \quad (19)$$

$$= \|\bar{p}(x, y) - \text{Tr}[P_{x,y} Q_z (\rho_{\text{sparse}} + \rho_{\text{faulty}})]/q_{z=0}\|_{l_1}, \quad (20)$$

where $q_{z=0} \equiv \text{Tr}[Q_z (\rho_{\text{sparse}} + \rho_{\text{faulty}})]$ is the probability to postselect the null syndrome measurements. Moreover, the sparse and faulty sets are redefined such that under the postselection of $z=0$ the operators in the sparse set results in the correct probability distribution:

$$\bar{p}(x, y) \propto \text{Tr}[P_{x,y} Q_z \rho_{\text{sparse}}]/q_{z=0} \quad (21)$$

$$\equiv \beta \bar{p}(x, y). \quad (22)$$

Since we have

$$1 = \sum_{x,y} \text{Tr}[P_{x,z} Q_z \rho_{\text{sparse}}] / q_{z=0} \quad (23)$$

$$= \beta + \text{Tr}[Q_z \rho_{\text{sparse}} Q_z] / q_{z=0}, \quad (24)$$

we obtain $1 - \beta = \text{Tr}[Q_z \rho_{\text{sparse}} Q_z] / q_{z=0}$. Then we obtain a similar bound on the error between the ideal probability distribution and the postselected noisy probability distribution:

$$\Delta = \|(1 - \beta)\bar{p}(x, y) - \text{Tr}[P_{x,y} Q_z \rho_{\text{faulty}}] / q_{z=0}\|_{l_1} \quad (25)$$

$$\leq (1 - \beta) + \|Q_z \rho_{\text{faulty}} Q_z\|_1 / q_{z=0} \quad (26)$$

$$\leq 2\|\rho_{\text{faulty}}\|_1 / q_{z=0}. \quad (27)$$

To proceed further calculation, we assume that \mathcal{E}_k is a CPTP map, i.e., the noise \mathcal{N}_k is a stochastic noise. In this case, we have $\|\mathcal{E}_k\| = \epsilon_k$. Moreover, since both ρ_{sparse} and ρ_{faulty} are density matrices, the postselection probability $q_{z=0}$ is lower-bounded:

$$q_{z=0} > \text{Tr}[Q_z \rho_{\text{sparse}} Q_z] \quad (28)$$

$$> \text{Tr}[Q_z \prod_k (1 - \epsilon_k) \mathcal{U}(\rho_{\text{ini}})] \quad (29)$$

$$= \prod_k (1 - \epsilon_k). \quad (30)$$

Thus the error Δ is again upper bounded as follows:

$$\Delta < 2 \sum_{\{\eta_k\}|\text{faulty}} \left(\frac{\epsilon_k}{1 - \epsilon_k} \right)^{\eta_k}. \quad (31)$$

If the system is designed fault-tolerantly and if ϵ_k is smaller than a certain constant value, the r.h.s. of Eq. (31) is upper-bounded by an exponentially small value. Therefore we have the following postselected threshold theorem:

Theorem 1 (postselected threshold theorem)

Suppose noise is given as a stochastic one $\mathcal{N}_k = (1 - \epsilon_k)\mathcal{I} + \mathcal{E}_k$. If the noise strength ϵ_k is smaller than a certain threshold value, we can simulate a probability distribution $\bar{p}(x, y)$ of an arbitrary universal quantum computation (uniformly generated polynomial-time quantum circuits) with an exponentially small additive error by using postselected noisy probability distribution $p(x, y|z = 0)$.

(The proof has been shown already in the above.)

Furthermore, we can also show that simulation of $\bar{p}(x, y)$ with an exponentially small additive error is actually enough to show hardness of a sampling according to $p^{\text{noisy}}(x, y, z)$ (see also Ref. [19]):

Lemma 1 Let C_ω and $p_\omega(x, y)$ be a uniformly generated polynomial-time quantum circuit and its probability distribution, respectively. If there exists a noisy quantum circuit $\mathcal{U}^{\text{noisy}}$ of the size $N = \text{poly}(n, \kappa)$ with n being the size of C_ω such that

$$|p_\omega(x, y) - p(x, y|z = 0)| < e^{-\kappa}, \quad (32)$$

then weak classical simulation i.e., sampling according to $p(x, y, z)$ with the multiplicative error $c < \sqrt{2}$ is impossible unless the PH collapses to the third level.

Here weak classical simulation with a multiplicative error c means that the classical sampling of (x, y, \dots) according to the probability distribution $p^{\text{samp}}(x, y, \dots)$ that satisfies

$$(1/c)p(x, y, \dots) < p^{\text{samp}}(x, y, \dots) < cp(x, y, \dots), \quad (33)$$

Proof: A language L is in the class postBQP iff there exists a uniform family of postselected quantum circuits $\{C_\omega\}$ with a decision port x and a postselection port y such that

$$\text{if } \omega \in L, p_\omega(x|y = 0) \geq 1/2 + \delta \quad (34)$$

$$\text{if } \omega \notin L, p_\omega(x|y = 0) \leq 1/2 - \delta, \quad (35)$$

where δ can be chosen arbitrary such that $0 < \delta < 1/2$. Note that without loss of generality we can assume the probability to obtain $y = 0$ is bounded, $p_\omega(y = 0) > 2^{-6n-4}$ as shown in Ref. [19]. Now we have

$$|p(x|y = 0, z = 0) - p_\omega(x|y = 0)| \quad (36)$$

$$< \left| p(x, y|z = 0) \left(\frac{1}{p(y = 0|z = 0)} - \frac{1}{p_\omega(y = 0)} \right) \right|$$

$$+ \left| \frac{p(x, y|z = 0) - p_\omega(x, y)}{p_\omega(y = 0)} \right| \quad (37)$$

$$< \frac{2e^{-\kappa}}{p(y = 0|z = 0)p_\omega(z = 0)} + \frac{e^{-\kappa}}{p_\omega(z = 0)} \quad (38)$$

$$< \frac{2e^{-\kappa}}{(p_\omega(y = 0) - e^{-\kappa})p_\omega(y = 0)} + \frac{e^{-\kappa}}{p_\omega(y = 0)}. \quad (39)$$

Since $p_\omega(y = 0) > 2^{-6n-4}$, we can choose $\kappa = \text{poly}(n)$ such that $|p(x|y = 0, z = 0) - p_\omega(x|y = 0)| < 1/2$. The resultant size of the noisy quantum circuit is still polynomial in n . From the definition (robustness against the bounded error) of the class postBQP (as same as postBQP), the postselected noisy quantum circuit can decide problems in postBQP=PP (recall that we can freely choose $0 < \delta < 1/2$). Thus postselected quantum computation of such noisy quantum circuits is as hard as PP, and hence cannot be weakly simulated with the multiplicative error $c < \sqrt{2}$ unless the PH collapses to the third level.

□

We have considered an approximated sampling with an multiplicative error c in Lemma 1. Approximation with the constant multiplicative error imposes a stronger requirement on classical computers than the constant additive error with l_1 -norm [20, 21]. However, all imperfections including measurements are taken into \mathcal{N}_k on unitary gates \mathcal{U}_k . Since here we are interested whether or not the outputs of the actual noisy experimental device possess quantum supremacy or not, an exact sampling, i.e. $c = 1$, is still enough for our purpose. Note also that

strong simulation, i.e., a calculation of a probability distribution $p(x)$ for a given x , with a constant multiplicative error is too strong notion of classical simulation, and hence it is much harder than what the actual experimental device does. However, an exact weak simulation, in which we are interested, is what the actual experimental device does.

By combining Theorem 1 and Lemma 1, we obtain the following threshold theorem of quantum supremacy:

Theorem 2 (Threshold of Quantum Supremacy)

Suppose noise is given as a stochastic one $\mathcal{N}_k = (1 - \epsilon_k)\mathcal{I} + \mathcal{E}_k$ with $\|\mathcal{E}\|_\diamond = \epsilon_k$. Each unitary gate \mathcal{U}_k chosen from a universal set of gates is followed by such a noise \mathcal{N}_k . If the noise strength ϵ_k is smaller than a certain constant value, an efficient weak classical simulation of the sampling according to the noisy quantum circuit $p(x, y, z, \dots) = \text{Tr}[P_{x,y,z,\dots} \mathcal{U}^{\text{noisy}}(\rho_{\text{ini}})]$ is impossible unless the PH collapses to the third level.

The above theorem indicates that even noisy sampling using noisy quantum circuits can have a power to exhibit quantum supremacy against a classical simulation of them. One of the main benefit to employ the threshold theorem of quantum supremacy is that we can show quantum supremacy experimentally in a very noisy region where the noise strength is above the standard threshold, which we call a pre-threshold region. In the following, we apply the above theorem for two prototypical cases: concatenated fault-tolerant quantum computation [5–8] and topological fault-tolerant quantum computation with the surface codes [46–48].

IV. CASE I: CONCATENATED QUANTUM COMPUTATION

To obtain a further insight, we first consider a concatenated fault-tolerant quantum computation. Suppose each fault-tolerant logical gate at the concatenation level l consists of at most M logical gates of the level $(l-1)$. If we employ a quantum error correction code of a distance d , any of at most $t \equiv \lfloor d/2 \rfloor$ errors never causes a logical error. At a concatenation level l , the faulty operator is bounded by

$$\epsilon^{(l)} \equiv \sum_{r=t+1}^M \binom{M}{r} (\epsilon^{(l-1)})^r (1 - \epsilon^{(l-1)})^{M-r} \quad (40)$$

$$\leq C(\epsilon^{(l-1)})^{t+1}. \quad (41)$$

By considering the concatenation, $\|\rho_{\text{fauldy}}\|_1$ is bounded in terms of $\epsilon \equiv \max_k \epsilon_k$ as follows:

$$\|\rho_{\text{fauldy}}\|_1 < (C^{1/t} \epsilon)^{(t+1)^l} / C^{1/t}, \quad (42)$$

where l is the number of concatenation levels and chosen to be logarithm in the size of computation. The threshold value is given roughly by $1/C^{1/t}$. On the other hand if we apply the threshold theorem for quantum supremacy, the

faulty operator is bounded at each concatenation level as follows:

$$\epsilon^{(l)} = \sum_{r=d}^M \binom{M}{r} (\epsilon^{(l-1)})^r (1 - \epsilon^{(l-1)})^{M-r} \leq C'(\epsilon^{(l-1)})^d. \quad (43)$$

Similarly to the previous case, the threshold is given roughly given by $1/C'^{1/(d-1)}$. For example, let us take $d = 3$ and $t = 1$, and assume $M \gg 1$. In the leading order, $C \sim M^2/2$ and $C' \sim M^3/6$. The threshold of quantum supremacy $\epsilon_{\text{th}} \sim \sqrt{6}/M^{3/2}$ is improved from that $\epsilon_{\text{th}} \sim 2/M^2$ for universal quantum computation by a factor of $O(\sqrt{M})$.

V. CASE II: FAULT-TOLERANT QUANTUM COMPUTATION WITH THE SURFACE CODES

Next we will consider topologically protected fault-tolerant quantum computation with the surface code [46–49, 53] to obtain a practical threshold value for quantum supremacy. Quantum error correction using the surface code with imperfect syndrome measurements is governed on primal and dual cubic lattices (see Ref. [53] for a detailed review). Below we consider the primal cubic lattice only, by assuming error correction is done on primal and dual cubic lattices independently, which results in an underestimate of the threshold. Then, errors are assigned on edges of the cubic lattice as an error chain, and the errors are detected at the boundary of the error chain. If the error and recovery chains result in a topologically nontrivial cycle, the error correction fails. In the topologically protected region, the defects representing logical qubits are designed such that the nontrivial cycle consists of a connected chain of length longer than d . Around the singular qubit for non-Clifford operations, we have to take into account nontrivial cycles of length shorter than d too.

Let us first consider the phenomenological noise model, where the errors are distributed independently and identically on each edge with probability ϵ . Now ρ_{fail} is divided into two parts ρ_{top} and ρ_{sin} , which correspond to the errors in the topologically protected region and others originated from the errors around the singular qubits, respectively. Since we can postselect the null syndrome measurements, ρ_{top} is attributed from the errors on the connected chain of length longer than d (the error chain of length shorter than d always results in an erroneous syndrome in the topologically protected region, and hence is postselected). Therefore, we have

$$\|\rho_{\text{top}}\|_1 \leq \text{poly}(n) \sum_{l=d} C_l \left(\frac{\epsilon}{1-\epsilon} \right)^l, \quad (44)$$

where n is the size of the quantum computation, and $C_l < (6/5)5^l$ is the number of self-avoiding walks [54] of length l . Apparently, this converges to zero if $\epsilon/(1-\epsilon) <$

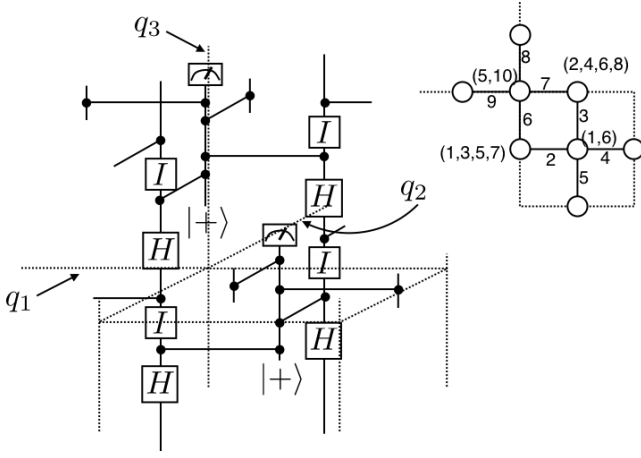


FIG. 1. The depth-8 circuit for syndrome measurements of the surface code. The top view is also shown right. An error is assigned on each edge with probabilities q_1 , q_2 , and q_3 independently. In addition, correlated errors occur with probabilities $q_{1,2}$, $q_{2,3}$, and $q_{3,1}$ on the two connected edges.

$1/5$. Note that while the resultant threshold $\epsilon = 0.167$ is somehow underestimated by the above analytical treatment, it is much higher than the standard threshold $0.0293 - 0.033$ [54–56] in the topologically protected region calculated from numerical simulations. Therefore, the protection around singular qubit becomes important. Around the singular qubits, the logical error is bounded by

$$\sum_{l=1}^d C'_l \left(\frac{\epsilon}{1-\epsilon} \right)^l, \quad (45)$$

where C'_l is the number of the self-avoiding walks that result in the logical errors of length shorter than d around the singular qubits (the error of length longer than d is taken in ρ_{topo}). In Ref. [19] (Tab. 1), C'_l is counted rigorously up to $l = 14$. This tells us that if $\epsilon/(1-\epsilon) < 0.134$ ($\epsilon < 0.118$) the amount of errors on each singular qubit becomes smaller than $0.146 = (1 - \sqrt{2}/2)/2$, the threshold of the magic state distillation [44, 45]. Therefore $\|\rho_{\text{sim}}\|_1$ converges to zero if $\epsilon < 0.118$. Accordingly, we have the threshold $\epsilon = 0.118$ for quantum supremacy with the surface code under the stochastic phenomenological noise model, which is much higher than the standard threshold $2.93\% - 3.3\%$ for universal fault-tolerant quantum computation.

Finally, we derive a noise threshold of quantum supremacy in the circuit-based noise model. Specifically, we employ a circuit shown in Fig. 1 for the syndrome measurements of the surface code. (Contrast to the circuit in Ref. [48], this circuit is not the lowest depth one. However, this setup is convenient to model the correlated errors.) We take the standard depolarizing noise

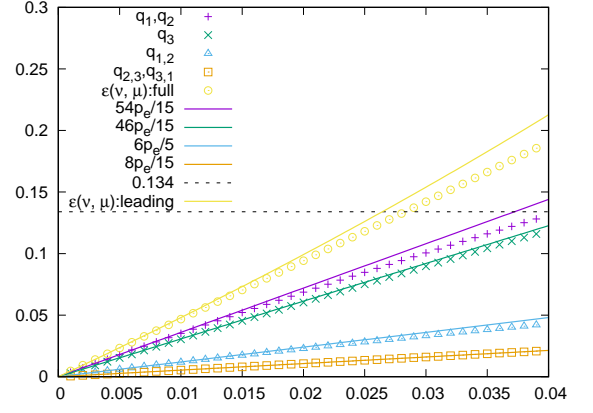


FIG. 2. The parameters for the error distribution, q_1 , q_2 , q_3 , $q_{1,2}$, $q_{2,3}$, and $q_{3,1}$, and the effective single-qubit error probability $\epsilon(v, \mu)$ are shown in the leading order (lines) and to all orders (points) as functions of p_e . The dashed line shows the threshold 13.4% for the effective single-qubit error probability.

for single- and two-qubit gates:

$$\mathcal{N}^{(1)} = [I] + \sum_{A=X,Y,Z} (p_1/3)[A] \quad (46)$$

$$\mathcal{N}^{(2)} = [I] + \sum_{A,B=X,Y,Z \setminus (A,B)=(I,I)} (p_2/15)[A \otimes B] \quad (47)$$

where we use the notation $[W]\rho \equiv W\rho W^\dagger$. The state preparations and measurements are followed and preceded by flipping the states in their bases with probabilities p_p and p_m , respectively. Then, the error distribution on the cubic lattice is characterized by single-qubit error probabilities q_1 , q_2 , and q_3 , and two-qubit correlated error probabilities $q_{1,2}$, $q_{2,3}$, and $q_{3,1}$, where the labels 1, 2 and 3 correspond to two space-like and one time-like axes. In the leading order, they are given by

$$q_1 = q_2 = 6\frac{4p_2}{15} + 3\frac{2p_1}{3}, \quad (48)$$

$$q_3 = 4\frac{4p_2}{15} + p_p + p_m, \quad (49)$$

$$q_{1,2} = 2\frac{4p_2}{15}, \quad (50)$$

$$q_{2,3} = q_{3,1} = 2\frac{4p_2}{15} + \frac{2p_1}{3}. \quad (51)$$

We also numerically evaluated q_1 , q_2 , q_3 and $q_{1,2}$, $q_{2,3}$, $q_{3,1}$ to all orders as shown in Fig. 2, which are in good agreement with the leading order evaluations but become slightly smaller than them by increasing p_e . Note that at the boundary and inside the defects a part of the gates for the syndrome measurements are not performed, and hence the actual error probabilities are smaller there. Let us define $\nu = \max\{q_1, q_2, q_3\}$ and $\mu = \max\{q_{1,2}, q_{2,3}, q_{3,1}\}$. Since we have a correlated error on the connected two edges with probability at most

μ , the probability of an error chain of the length l (in Eqs. (44) and (45)) is now replaced by

$$C_l \sum_{k=0}^{\lfloor l/2 \rfloor} \binom{\lfloor l/2 \rfloor}{k} 2^k \left(\frac{\nu}{1-\nu} \right)^{l-k} \left(\frac{\mu}{1-\mu} \right)^k \quad (52)$$

$$< C_l \left(\frac{\nu}{1-\nu} \right)^{l-\lfloor l/2 \rfloor} \left[\left(\frac{\nu}{1-\nu} \right) + \left(\frac{2\mu}{1-\mu} \right) \right]^{\lfloor l/2 \rfloor} \quad (53)$$

Equation (52) reads as follows. The edges on the chain of the length l are labeled from 1 to l . For $k = 0, 1, \dots, \lfloor l/2 \rfloor$, k correlated errors are chosen from the $\lfloor l/2 \rfloor$ edges labeled by odd numbers. Then, each of the chosen edges can correlate two neighboring even number edges.

Equation (53) can be viewed as a stochastic phenomenological noise model with an effective single-qubit error probability $\epsilon(\nu, \mu)$,

$$\epsilon(\nu, \mu) \equiv \left(\frac{\nu}{1-\nu} \right)^{1/2} \left[\left(\frac{\nu}{1-\nu} \right) + \left(\frac{2\mu}{1-\mu} \right) \right]^{1/2} \quad (54)$$

Therefore, similarly to the previous argument, if $\epsilon(\nu, \mu) < 0.134$, $\|\rho_{\text{faultry}}\|_1$ decreases exponentially. By setting $p_e = p_1 = p_2 = p_p = p_m$, $\nu = 54p_e/15$ and $\mu = 6p_e/5$ in the leading order, which results in the threshold $p_e = 2.64\%$. If employ the all-order evaluations as shown in Fig. 2, the threshold is slightly improved to $p_e = 2.84\%$. The obtained thresholds for quantum supremacy are again much higher than the standard threshold 0.75% for universal fault-tolerant quantum computation. Note that here the errors on the singular qubits are far overestimate. Some errors on the singular qubits have correlation with the errors on the dual cubic lattice, and hence can be post-selected. If we consider the correlation between the errors on the primal and dual cubic lattices, the threshold of quantum supremacy would be further improved.

While the standard threshold 0.75% is limited by the threshold in the topologically protected region, the post-selected threshold of quantum supremacy 2.84% is determined purely from the limitation of magic state distillation. Namely, quantum supremacy in the noisy quantum circuits is originated from distillability of the magic state.

VI. CONCLUSION AND DISCUSSION

Here we have derived the threshold theorem of quantum supremacy with noisy quantum circuits in the pre-threshold region. While we employed noisy but universal set of gates here, it would be interesting to apply the theorem for non-universal quantum computational models such as BosonSampling, IQP, and DQC1 (see Ref. [19] in the case of noisy commuting circuits). In the case of BosonSampling, if we want to show universality under postselection, we have to take non-deterministic gates into account. Fault-tolerant linear optical quantum computation [26, 57–61] would be employed even in this case. On the other hand, in the case of DQC1 [24, 25], the number of measurement ports seems to be too small to perform fault-tolerant quantum computation. This problem might be avoided by employing polynomially many measurements, but still it is quite nontrivial to construct a fault-tolerant circuit by using completely randomized ancilla states and postselection. Contrast to BosonSampling and IQP, the output of DQC1, the normalized trace of a unitary operator, appears ubiquitously in physics and has a lot of applications, such as spectral density estimation [23], testing integrability [62], calculation of fidelity decay [63], and approximation of the Jones and HOMFLY polynomials [64–66]. Fault-tolerance of quantum supremacy in DQC1 is an important open problem.

We have considered hardness of an exact (or a constant multiplicative approximation) weak classical simulation of the noisy quantum circuits to know whether or not the outputs of the actual experimental device possess quantum supremacy. It would be interesting to see how noise tolerance changes if we change the notion of approximation to the additive one with l_1 -norm [20, 21, 42], which provides an advantage to classical computers making their quantum targets relaxed. Is there still surviving quantum supremacy of pre-threshold noisy quantum circuits even in such a setting?

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